UNIT - IV

POISSON PROCESS :

It is an example for a s.p with discrete state space and continuous parametric space.

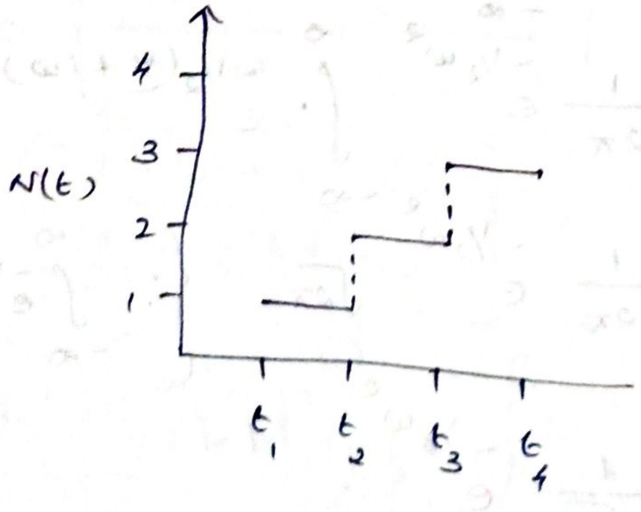
Example :

(i) No. of incoming telephone calls at a telephone booth on $[0,t]$ interval

(ii) Arrival of customer for a service at a counter on $[0,t]$ interval

(iii) Occurance of an accident at a certain place on $[0,t]$ interval.

Let $N(t)$ denotes the total no. of occurance of the events in an interval of duration 't'. It may be given in the following graph



If the r.v $N(t)$ assumes the value 'i', then the prob. of the event $P[N(t)=i]$ is denoted by $P_i(t)$, where

$P_i(t)$ is a fn. of 't' such that $\sum_{i=0}^{\infty} P_i(t) = 1$, then $P_i(t)$ is a prob. ~~dens~~ dist. fn. of $N(t)$ for every value of 't'.

Definition:

If $\{N(t), t \geq 0\}$ is assumed as a counting process with $N(0) = 0$, $N(t)$ is said to be a poisson process. Eg: any 2

postulates (or) Assumptions:

With respect to the poisson process the following assumptions are made

(i) The increments are all independent (ie) for the time point $t_1, t_2, \ldots, t_n \ni$ $t_1 \leq t_2 \leq \ldots \leq t_n$.

$x(t_1)$, $x(t_2) - x(t_1)$, $x(t_3) - x(t_2)$, ... are independent.

(ii) Transition prob. satisfy the stationary prob. (ie) the dist. of $x(t) - x(s)$ for $s < t$ depends only on the time interval but not on s the time point where we started.

(iii) Prob. that atleast one event happen in the small interval of time h, $h > 0$ is given by

$$P_1(h) = \lambda h + o(h)$$

where $\frac{o(h)}{h} \to 0$ as $h \to 0$

Prob. that the more than one event happens in the small interval is $o(h)$, $h > 0$.

Derivation of Poisson process:

Let $P_m(t) = P[X(t) = m]$

which gives the prob. that 'm' events happen in the interval $[0, t]$

By postulates,

$$P_1(h) = \lambda h + o(h)$$

(i) ie prob. that the happening of atleast one event in the interval of time 'h'

(ii) prob. that no events happened in the interval 'h' is

$$P_0(h) = 1 - P(h)$$

(iii) prob. that 2 (or) more events happened in the interval of time 'h' is

$$\sum_{m=2}^{\infty} P_m(h) = 0.$$

Now consider $P_m(t) = 0$ for $m = 0, 1, \ldots$

for $h > 0$, the small interval

$$P_0(t+h) = P_0(t) \cdot P_0(h) \; ; \; h > 0$$

$$= P_0(t)\left[1 - P(h)\right]$$

$$= P_0(t)\left[1 - \lambda h - o(h)\right]$$

$$= P_0(t) - \lambda h P_0(t) - o(h) \cdot P_0(t)$$

$$P_0(t+h) - P_0(t) = -\lambda h P_0(t) - o(h) P_0(t)$$

Dividing by 'h' and $h \to 0$

$$\frac{P_0(t+h) - P_0(t)}{h} = \frac{-\lambda \cancel{h} P_0(t)}{\cancel{h}} - \frac{o(h) P_0(t)}{h}$$

$$P_0'(t) = -\lambda P_0(t) \quad \text{as } h \to 0$$

which is a differential equ. whose solu. is given by

$$P_0(t) = C . e^{-\lambda t}$$

where C is to be determined from ~~initial~~ initial conditions.

Put $t = 0$,

$$P_0(0) = C . e^{-\lambda(0)}$$

$$P_0(0) = C$$

$$\boxed{1 = C} \quad \{\because P_0(0) = 1\}$$

$$\therefore P_0(t) = e^{-\lambda t} \quad \text{—} ①$$

Now consider,

$$P_m(t+h) = P_0(h) P_m(t) + P_1(h) P_{m-1}(t) + P_2(h) P_{m-2}(t) + \dots$$

$$= P_0(h) P_m(t) + P_1(h) P_{m-1}(t) + \sum_{i=2}^{\infty} P_i(h) P_{m-i}(t)$$

$$= [1 - \lambda h - o(h)] P_m(t) + [\lambda h + o(h)] P_{m-1}(t) + 0$$

$$P_m(t+h) = P_m(t) - \lambda h P_m(t) - o(h) P_m(t) + \lambda h P_{m-1}(t) + o(h) P_{m-1}(t)$$

$$P_m(t+h) - P_m(t) = -\lambda h P_m(t) - o(h) P_m(t) + \lambda h P_{m-1}(t) + o(h) P_{m-1}(t)$$

Dividing by h & $h \to 0$

$$\frac{P_m(t+h) - P_m(t)}{h} = \frac{-\lambda h P_m(t)}{h} + \frac{\lambda h\, P_{m-1}(t)}{h}$$

$$P_m'(t) = -\lambda\left[P_m(t) - P_{m-1}(t)\right] \text{ — } ②$$

which is a differential equ. This equ. can be solved either by Induction method (or) by prob. Generating method.

(i) Induction method:

$$\text{Let, } Q_m(t) = P_m(t) \cdot e^{\lambda t} \rightarrow (1)$$

$$P_m(t) = Q_m(t) \cdot e^{-\lambda t} \rightarrow (2)$$

From (1),

$$t = 0$$

$$Q_m(0) = P_m(0) \cdot e^{\lambda(0)} = 0 \rightarrow (3)$$

$$m = 0 \Rightarrow Q_0(0) = P_0(0) \cdot e^{\lambda(0)} = 1 \rightarrow (4)$$

Diff equ.(2) w.r.to 't'.

$$\frac{\partial P_m(t)}{\partial t} = P_m'(t) = Q_m'(t) \cdot e^{-\lambda t} - \lambda Q_m(t) \cdot e^{-\lambda t} \rightarrow (5)$$

But

$$P_m'(t) = -\lambda\left[P_m(t) - P_{m-1}(t)\right]$$

$$= -\lambda\left[Q_m(t)\, e^{-\lambda t} - Q_{m-1}(t)\, e^{-\lambda t}\right] \rightarrow (6)$$

from (5) and (6),

$$Q_m'(t) \cdot e^{-\lambda t} - \lambda Q_m(t)\, e^{-\lambda t} = -\lambda Q_m(t)\, e^{-\lambda t} + \lambda Q_{m-1}(t) \cdot e^{-\lambda t}$$

$$Q_m'(t)\, e^{-\lambda t} = \lambda Q_{m-1}(t)\, e^{-\lambda t} \rightarrow (7)$$

Put $m = 1$ in equ (7),

$$Q_1'(t) = \lambda Q_0(t) = \lambda(1) = \lambda \quad (\because Q_0(t) = 1)$$

Integrating 't' both sides, in equ. (7).

$$Q_1(t) = \lambda t + C$$

To determine 'c', consider $Q_1(0) = 0 + C$

Put $\boxed{t = 0}$ $\quad 0 = 0 + C \Rightarrow \boxed{C = 0}$

$$\boxed{\therefore Q_1(t) = \lambda t}$$

$$P_1(t)\, e^{\lambda t} = \lambda t \text{ from (1)}$$

$$P_1(t) = e^{-\lambda t}(\lambda t)$$

When $m = 2$, in equ. (7)

$$Q_2'(t) = \lambda Q_1(t) = \lambda . \lambda t$$

Integrating on both sides,

$$Q_2(t) = \lambda^2 \frac{t^2}{2} + c$$

$c = 0$ as in previous case,

$$\Rightarrow Q_2(t) = \frac{(\lambda t)^2}{2}$$

$$P_2(t)\, e^{\lambda t} = \frac{(\lambda t)^2}{2} \text{ from eqn (1)}$$

$$P_2(t) = \frac{e^{-\lambda t}(\lambda t)^2}{2!}$$

$$P_m(t) = \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

which is poisson process.

(ii) Generating function method:

Consider,

$$P_m'(t) = -\lambda\left[P_m(t) - P_{m-1}(t)\right]$$

$$\frac{\partial P_m(t)}{\partial t} = -\lambda\left[P_m(t) - P_{m-1}(t)\right]$$

Multiplying on both sides by s^m and taking summation over 'm',

$$\frac{\partial \sum_{m=1}^{\infty} P_m(t)\, s^m}{\partial t} = -\lambda\left[\sum_{m=1}^{\infty} P_m(t)\, s^m - \sum_{m=1}^{\infty} P_{m-1}(t)\, s^m\right] \hookrightarrow (1)$$

Consider the propability generating function,

$$P(s,t) = \sum_{m=0}^{\infty} P_m(t)\, s^m$$

$$= P_0(t)\, s^0 + \sum_{m=1}^{\infty} P_m(t)\, s^m$$

$$\Rightarrow \sum_{m=1} P_m(t)\, s^m = P(s,t) - e^{\lambda t} \rightarrow (2)$$

Consider,

$$\sum_{m=1} P_{m-1}(t)\, s^m = s.\sum_{m=1} P_{m-1}(t)\, s^{m-1}$$

$$= s.p(s,t) \rightarrow (3).$$

Diff. w.r. to 't',

From(2)

$$\frac{\partial \sum_{m=1} P_m(t)\, s^m}{\partial t} = P'(s,t) + \lambda e^{-\lambda t} \rightarrow (4)$$

using ① & ④,

$$p'(s,t)+\lambda e^{-\lambda t} = -\lambda\left[\Sigma P_m(t)s^m - \Sigma P_{m-1}(t)s^m\right]$$

$$= -\lambda\left[P(s,t) - e^{-\lambda t} - sP(s,t)\right]$$

{using ② & ③}

$$p'(s,t)+\cancel{\lambda e^{-\lambda t}} = -\lambda P(s,t) + \cancel{\lambda e^{-\lambda t}} + \lambda s P(s,t)$$

$$P'(s,t) = \lambda P(s,t)\left[s-1\right] \text{ ——— } ⑤$$

which is a differential equ. whose solu is

$$P(s,t) = C\cdot e^{\lambda(s-1)t}$$

To find 'c'

consider, $P(s,t) = \sum_{m=0} P_m(t)\, s^m$

$$= P_0(t)s^0 + \sum_{m=1} P_m(t)s^m$$

$$P(s,0) = P_0(0)s^0 + 1$$

$$= 1 \text{ ——— } ⑥$$

Now consider,

$$P(s,t) = c\, e^{\lambda(s-1)t}$$

$$= c\cdot 1 \text{ ——— } ⑦$$

From ⑥ & ⑦, c = 1.

Thus $P(s,t) = e^{\lambda(s-1)t}$

$$= e^{\lambda s t}\, e^{-\lambda t}$$

which can be written as

$$\sum_{m=0} P_m(t) s^m = e^{\lambda s t} . e^{-\lambda t}$$

$$= e^{-\lambda t} \sum_{m=0} \frac{(\lambda t)^m s^m}{m!}$$

$$\left\{ \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \right.$$
$$= \sum_{r=0} \frac{x^r}{r!}$$

consider the co-eff. of s^m

$$P_m(t) = \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

which is the required poisson proces

Properties of Poisson Process :
[Application (or) features]

(i) The sum of two poisson process is also a poisson process.

$\{X(t)\}$ is a poisson process

$$Q_{X(t)}(s) = E\left[e^{is\,X(t)}\right]$$
$$= e^{\lambda t[e^{is}-1]}.$$

$\{X_1(t)\}$ and $\{X_2(t)\}$ are two poisson process with parameter λ_1, λ_2 respectively.

Consider,

$$Q_{x_1(t)+x_2(t)}(s) = E\left[e^{is[x_1(t)+x_2(t)]}\right].$$

$$= E\left[e^{is\,x_1(t)}\right] \cdot E\left[e^{is\,x_2(t)}\right].$$

$$= e^{\lambda_1 t[e^{is}-1]} \; e^{\lambda_2 t[e^{is}-1]}.$$

$$= e^{(\lambda_1+\lambda_2)t\,[e^{is}-1]}$$

$= e^{\lambda t(e^{is}-1)}$ where $\lambda = \lambda_1 + \lambda_2$ which is similar to the c.f. of poisson process.

(ii) Binomial distn/- in relation with poisson process:

$\{x_1(t)\}$ is a poisson process with parameter λ_1.

$\{x_2(t)\}$ is a poisson process with parameter λ_2.

For an $m<n$, consider,

$$P[x_1=m / x_1+x_2=n] = \frac{P[x_1=m \cap x_1+x_2=n]}{P[x_1+x_2=n]}$$

$$= \frac{P[x_1=m \cap x_2=n-m]}{P[x_1+x_2=n]}$$

$$= \frac{P[x_1=m] \cdot P[x_2=n-m]}{P[x_1+x_2=n]}$$

$$= \frac{\frac{e^{-\lambda_1 t}(\lambda_1 t)^m}{m!} \cdot \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-m}}{(n-m)!}}{\frac{e^{-(\lambda_1+\lambda_2)t}[(\lambda_1+\lambda_2)t]^n}{n!}}$$

$$= \frac{n!}{m!(n-m)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}$$

Let $\frac{\lambda_1}{\lambda_1+\lambda_2} = p$, $\frac{\lambda_2}{\lambda_1+\lambda_2} = q$

$$= \binom{n}{m} p^m q^{n-m}$$

which is the p.d.f. of binomial distn/-

(iii) $x(t)$ is a poisson r.v, then mean of

$$x(t) = E[x(t)]$$

$$E[x(t)] = \sum_{n=0}^{\infty} x(t) \cdot P_i(t)$$

$$= \sum_{n=0}^{\infty} x(t) \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n. \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \cancel{n}. \frac{e^{-\lambda t}(\lambda t)^n}{\cancel{n}(n-1)!}$$

$$= (\lambda t)e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t \, e^{-\lambda t} \cdot e^{\lambda t}$$

$$E[x(t)] = \lambda t$$

$$\therefore \ Var[x(t)] = E[x(t)^2] - [E(x(t))]^2$$

$$E[x(t)^2] = \sum x(t)^2 \, p_i(t)$$

$$= \sum_{n=0}^{\infty} x(t)^2 \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n^2 \frac{e^{-\lambda t} \lambda t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{n}{n!} + \frac{n(n-1)}{n!} \right] e^{-\lambda t} (\lambda t)^n$$

$$= \sum_{n=0}^{\infty} \frac{n}{n(n-1)!} + \frac{n(n-1)}{n(n-1)(n-2)} e^{-\lambda t} (\lambda t)^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{(n-1)!} + \frac{1}{(n-2)!} \right] e^{-\lambda t} (\lambda t)^n$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{(n-2)!}$$

$$= \lambda t + (\lambda t)^2$$

$$var[x(t)] = \lambda t + (\lambda t)^2 - (\lambda t)^2$$

$$var[x(t)] = \lambda t$$

(iv) C.F. of the poisson process:

$$\phi_{x(t)}(s) = E\left[e^{is\, x(t)}\right]$$

$$= e^{\lambda t}\left[e^{is} - 1\right].$$

Birth and Death process:

It is one of the important classes of Markov process. Its state space is countable such that (0, 1, 2, ...) $P_n(t)$. Let $x(t)$ denote the pop/- size at time 't'. It can be increase by birth, decrease by death. Multiple birth (or) multiple death are ruled out. They are assumed as zero. The birth and death rates depends on 't'. But certainly depends pop size. If they do not depend on 't', we get a time homogenous transition probability.

$$P_{ij}(t) = P\{x(t) = j / x(0) = i\}$$

$$= P\{x(t+u) = j / x(u) = i\} \ \forall u.$$

Assumptions:

$$\left.\begin{array}{l}
\text{(i) } P\{x(t+h) = k-1 / X(t) = k\} = \mu_k h + o(h) \text{ for } k = 1, 2, \ldots \\
\text{(ii) } P\{x(t+h) = k+1 / x(t) = k\} = \lambda_k h + o(h) \\
\text{(iii) } P\{x(t+h) = k / x(t) = k\} = 1 - \lambda_k h - \mu_k h + o(h) \\
\text{(iv) } P\{|x(t+h) - x(t)| > 1 / x(t) = k\} = o(h).
\end{array}\right\} \to (1)$$

The stochastic process x(t) is called Birth and death process if it satisfies the assumption given in (1).

Thus x(t) is a markov process with state space (0,1,2,...) and intensity rates are,

$$q_{jk} = \begin{cases} \mu_j & ; \text{ for } k = j-1 \\ \lambda_j & ; \text{ for } k = j+1 \\ -(\lambda_j + \mu_j) & ; \text{ for } k = j. \end{cases} \rightarrow (2)$$

for $j = 1, 2$ and $\mu_0 = 0$

for $j = 0$; $k = 1$,

$$q_{01} = \lambda_0$$

for $j = 0$, $k = 0$;

$$q_{00} = -\lambda_0 \quad [\because \mu_0 = 0].$$

From eq. (2) the forward kolmogorov differential eq. of B and D. process are obtained as follows.

$$P_j'(t) = \{-(\lambda_j + \mu_j) P_j(t) + \lambda_{j-1} P_{j-1}(t) + \mu_{j+1} P_{j+1}(t); \text{ for } j \geq 1\} \rightarrow (3)$$

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t).$$

Derivation of (3):

Let us consider three time points $0, t, t+h$, such that $0 < t < t+h$.

$$(0, t+h) = (0, t) \cup (t, t+h)$$

The no. of popl. at time $t+h$,

$$P_{ij}(t+h) = P_{ij}(t) . P\{x(t+h) = j / x(t) = j\} + P_{i,j-1}(t)$$

$$\cdot \{P\{x(t+h) = j / x(t) j-1\} + P_{i,j+1}(t)$$

$$P\{x(t+h) = j / x(t) = j+1\}$$

$$P_{ij}(t+h) = \sum_{r=j,j-1,j+1} P_{ir}(t) P_{rj}(h)$$

$$= P_{ij}(t) P_{jj}(h) + P_{i,j-1}(t) P_{j-1,j}(h)$$

$$+ P_{i,j+1}(t) P_{j+1,j}(h).$$

$$= P_{ij}(t)[1 - (\lambda_j + \mu_j)h + P_{i,j-1}(t) \lambda_{j-1}(h)$$

$$+ P_{i,j+1}(t) \mu_{j+1}(h) + o(h)$$

$$= P_{ij}(t) - P_{ij}(t)(\lambda_j + \mu_j)h + \lambda_{j-1}(h). P_{i,j-1}(t)$$

$$+ \mu_{j+1}(h) P_{i,j+1}(t) + o(h).$$

$$P_{ij}(t+h) - P_{ij}(t) = -P_{ij}(t)(\lambda_j + \mu_j)(h) + \lambda_{j-1}(h) P_{i,j-1}(t)$$

$$+ \mu_{j+1}(h) P_{i,j+1}(t) + o(h).$$

Dividing by h and taking limit h → 0.

$$\lim_{h \to 0} \left[\frac{P_{ij}(t+h) - P_{ij}(t)}{h} \right] = -(\lambda_j + \mu_j) P_{ij}(t) + \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) + \lim_{h \to 0} \left[\frac{o(h)}{h} \right].$$

$$P_{ij}'(t) = -(\lambda_j + \mu_j) P_{ij}(t) + \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t)$$

at $t = 0$,

$$P_{ij}'(0) = -(\lambda_j + \mu_j) P_{ij}(0) + \lambda_{j-1} P_{i,j-1}(0) + \mu_{j+1} P_{i,j+1}(0)$$

Let, $P_j(t) = P_{ij}(t)$, suppressing the suffix 'i'.

Then we get,

$$P_j'(t) = -(\lambda_j + \mu_j) P_j(t) + \lambda_{j-1} P_{j-1}(t) + \mu_{j+1} P_{j+1}(t)$$

At $j = 0$,

$$P_0'(t) = -(\lambda_0 + \mu_0) P_0(t) + \lambda_{-1} P_{-1}(t) + \mu_1 P_1(t).$$

Since initial condition, $P_j(0) = \begin{cases} 0 ; & j = 1, 2, \ldots \\ 1 ; & j = 0. \end{cases}$

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad \because \mu_0 = 0.$$

The forward kolmogorov partial dif/. eqn. for B & D process.

$$P_j'(t) = -(\lambda_j + \mu_j) P_j(t) + \lambda_{j-1} P_{j-1}(t) + \mu_{j+1} P_{j+1}(t).$$

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t).$$

The soln/. $P_j(t)$ can be obtained by solving the forward kolmogorov partial diff/. eqn. subject to the initial condition.

Uses of B and D process :

B and D process serves as a good model in the study of different disciplines of queueing theory.

Let x(t) be the no. of persons waiting or being served for service at a counter at time 't'. This number may increase due to an arrival & decrease due to service completion.

If λ_j denotes the arrival rate, μ_j denote the service rate and j persons on the queue, then the process x(t) can be modelled as a B and D process with assumptions given in (1).

Note:

For different values of λ_j, μ_j differential models are formed. If $\lambda_j = j\lambda$ and $\mu_j = 0$, then then B and D process becomes linear birth process. If $\lambda_j = \lambda$ and $\mu_j = 0$, then the B and D process becomes immigration process. If $\lambda_j = 0$, $\mu_j = \mu$ then then the B and D process is called as immigration, Emmigration process. If $\mu_j = j\mu$ and $\lambda_j = 0$, then B and D process is called linear death process. If $\lambda_j = \lambda_j$ and $\mu_j = 0$, then B & D process becomes pure birth process. If $\lambda_j = 0$ and $\mu_j = \mu_j$, then B and D process becomes pure death process. The pure birth process becomes Yule - furry process if $\lambda_j = j\lambda$.

Pure - birth process:

Let $x(t)$ denotes the popl size at time 't'.

$x(t)$ can increase by birth but multiple birth are not included in the process.

Further $x(t)$ cannot decrease by death as well as migration.

(ie) $\mu_k = 0$, then the forward kolmogorov differential eqn's can be derived as follows.,

Let $x(0) = 0$, $P_n(t) = P\{x(t) = n / x(0) = 0\}$

$$P_n(t+h) = P_0(h) P_n(t) + P_1(h) P_{n-1}(t) + \sum_{i=2} P_i(h) P_{n-i}(t)$$

$$= [1 - \lambda_n h + o(h)] P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h) + \dots$$

$$\lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h} = -P_n(t) \lambda_n + P_{n-1}(t) \lambda_{n-1} + \lim_{h \to 0} \frac{o(h)}{h}$$

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) + 0 \quad ——— ①$$

At $n=0$,

$$P_0'(t) = -\lambda_0 P_0(t) \quad ②$$

equ ① & ② are called kolmogorov partial diff. equ. of the pure birth process.

From ②,

$$P_0'(t) + \lambda_0 P_0(t) = 0$$

$$P_0(t)\, e^{\int \lambda_0 dt} = k \Rightarrow P_0(t)\, e^{\lambda_0 t} = k$$

At $t=0$,

$$P_0(0)\, e^{\lambda_0 (0)} = k$$

$$\Rightarrow k = 1 \quad \{\because P_0(0) = 1\}$$

$$\therefore P_0(t) = e^{-\lambda_0 t} \quad ③$$

$$P_0(t)\, e^{\int \lambda_0 dt} = k$$

diff. both sides

$$P_0'(t)\, e^{\lambda_0 \int dt} + P_0(t)\, e^{\lambda_0 \int d}$$

$$e^{\lambda_0 \int dt}\left[P_0'(t) + P_0(t)\lambda_0\right] =$$

$$P_0'(t) + \lambda_0 P_0(t) = 0$$

Put $n=1$ in equ ①, we get

$$P_1'(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t)$$

$$= -\lambda_1 P_1(t) + \lambda_0 e^{-\lambda_0 t} \quad [\text{using } ③]$$

$$P_1'(t) + \lambda_1 P_1(t) = \lambda_0 e^{-\lambda_0 t}$$

$$P_1(t)\, e^{\int \lambda_1 dt} = \int \lambda_0 e^{-\lambda_0 t} e^{\int \lambda_1 dt} dt + c$$

$$P_1(t)\, e^{\lambda_1 t} = \int \lambda_0 e^{-\lambda_0 t} e^{\lambda_1 t} dt + c.$$

$$= \lambda_0 \int e^{-(\lambda_0 - \lambda_1)t} dt + c.$$

$$= \lambda_0 \frac{e^{-(\lambda_0 - \lambda_1)t}}{-(\lambda_0 - \lambda_1)}$$

$$P_1(t)e^{\lambda_1 t} = \frac{\lambda_0}{\lambda_1-\lambda_0} e^{-(\lambda_0-\lambda_1)t} + c.$$

At $t = 0$

$$P_1(0)\, e^{\lambda_1(0)} = \left(\frac{\lambda_0}{\lambda_1-\lambda_0}\right) e^{-(\lambda_0-\lambda_1)(0)} + c.$$

$$0 \cdot 1 = \frac{\lambda_0}{\lambda_1-\lambda_0} + c$$

$$c = \left(-\frac{\lambda_0}{\lambda_1-\lambda_0}\right)$$

$$P_1(t)e^{\lambda_1 t} = \left(\frac{\lambda_0}{\lambda_1-\lambda_0}\right) e^{-(\lambda_0-\lambda_1)t} - \left(\frac{\lambda_0}{\lambda_1-\lambda_0}\right)$$

$$P_1(t) = \left(\frac{\lambda_0}{\lambda_1-\lambda_0}\right) e^{-\lambda_0 t + \cancel{\lambda_1 t} - \cancel{\lambda_1 t}} - \left(\frac{\lambda_0}{\lambda_1-\lambda_0}\right) e^{-\lambda_1 t}$$

$$= \left(\frac{\lambda_0}{\lambda_1-\lambda_0}\right)\left[e^{-\lambda_0 t} - e^{-\lambda_1 t}\right] \text{ ——— } ④$$

$$P_k(t) = \lambda_{k-1}\, e^{-\lambda_k t} \int_0^t e^{\lambda_k x} P_{k-1}(x)\,dx \text{ ——— } ⑤$$

for $k = 1, 2, \ldots$

which is called that soln. of pure birth process. $P_k(t)$ can be evaluate for $k = 1, 2, \ldots$ by Iteration.

Linear Birth process : (Yule - furry process)

when $\lambda_n = n\lambda$, $x(0) = 1$, then the pure birth process is called as Yule furry process (Linear Birth process).

The partial diff. equ. of the Yule - furry process can be obtained by subst. $\lambda_n = n\lambda$ in equ ①

$$\left.\begin{aligned}(\text{ie})\quad P_n'(t) &= -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)\\ &= -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)\\ &= -\lambda\left[nP_n(t) - (n-1)P_{n-1}(t)\right]\end{aligned}\right\} \text{——} (*)$$

Suppose the initial condition is

$$P_i(0) = \begin{cases} 0 & ; \ i \neq 1 \\ 1 & ; \ i = 1 \end{cases}$$

(ie) The process started with only one member at time $t = 0$.

when $n = 1$,

$$P_1'(t) = -\lambda P_1(t)$$

$$P_1'(t) + \lambda P_1(t) = 0$$

$$P_1(t)\, e^{\int \lambda dt} = k$$

$$P_1(t)\, e^{\lambda t} = k$$

At $t = 0$,

$$P_1(0)\, e^{\lambda(0)} = k \Rightarrow k = 1 \quad \{\text{ie } P_1(0) = 1\}$$

$\therefore P_1(t)\,e^{\lambda t} = 1$

$$P_1(t) = e^{-\lambda t} \text{ ——— } (**)$$

when $n = 2$,

$P_2'(t) = -2\lambda P_2(t) + \lambda P_1(t)$

$P_2'(t) + 2\lambda P_2(t) = \lambda P_1(t)$

$= \lambda e^{-\lambda t}$ (from $*$)

$P_2(t)\,e^{\int 2\lambda dt} = \int \lambda e^{-\lambda t} e^{\int 2\lambda dt}\,dt + c.$

$P_2(t)\,e^{2\lambda t} = \lambda \int e^{-\lambda t} e^{2\lambda t}\,dt + C$

$= \lambda \int e^{\lambda t}\,dt + C$

$= \cancel{\lambda}\dfrac{e^{\lambda t}}{\cancel{\lambda}} + C$

At $t = 0$,

$P_2(0)\,e^{2\lambda(0)} = 1 + C$ $\left\{\text{Let } P_2(0) = 0\right\}$

$0 = 1 + C$

$\Rightarrow C = -1$

$P_2(t)\,e^{2\lambda t} = e^{\lambda t} - 1$

$P_2(t) = \dfrac{e^{\lambda t} - 1}{e^{2\lambda t}}$

$= (e^{\lambda t} - 1)e^{-2\lambda t} = e^{\lambda t - 2\lambda t} - e^{-2\lambda t}$

$= e^{-\lambda t} - e^{-2\lambda t}$

$= e^{-\lambda t}\left[1 - e^{-\lambda t}\right]$

|||ly, $P_n(t) = e^{-\lambda t}\left[1 - e^{-\lambda t}\right]^{n-1}$; for $n \geq 1$

$= pq^{n-1}$

where $p = e^{-\lambda t}$; $q = 1 - e^{-\lambda t}$

$$P_n(t) = pq^{n-1}$$

which is called as Geometric dist^n.

The prob. generating fn. of the Geometric dis^l. is

$$P(s,t) = \sum_{n=1}^{\infty} P_n(t)\, s^n$$

$$= \sum_{n=1}^{\infty}\left[e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}\right] s^n$$

$$= \frac{se^{-\lambda t}}{1 - s(1 - e^{-\lambda t})}$$

$P(s,t) = \frac{sp}{1-sq}$

$E(x) = 1/p$

$V(x) = q/p^2$

which is the pdf of Yule-furry process.

Mean of the process $= \frac{1}{e^{-\lambda t}} = e^{\lambda t}$

Variance of the process $= \frac{1 - e^{-\lambda t}}{\left(e^{-\lambda t}\right)^2}$

$$= \left(1 - e^{-\lambda t}\right) e^{2\lambda t}$$

$$= e^{2\lambda t} - e^{\lambda t}$$

$$= e^{\lambda t}\left[e^{\lambda t} - 1\right]$$

Pure Death Process:

Pure Birth process can increase the popn. by jumps of unit magnitude. when the popn. can decrease over time, it may be modelled as a pure Death process.

Assumptions:

(i) popn. can be decrease by jumps of unit magnitude.

(ii) Inter death times are exponentially distributed

(iii) The popn. size at time 't' is $\{x(t), t \geq 0\}$ such that it is a Markov process with a stationary transition prob. fn. satisfying

$$\left.\begin{array}{l} (i)\ P\{x(t+h) = k / x(t) = k\} = 1 - \mu_k h + o(h) \\ (ii)\ P\{x(t+h) = k-1 / x(t) = k\} = \mu_k h + o(h) \\ (iii)\ P\{x(t+h) < k-1 / x(t) = k\} = o(h) \end{array}\right\} \text{———} ①$$

The initial condn. states that $x(t)\Big|_{t=0} = x(0) = N$ and the state space of the process $\{x(t)\}$ is finite $\ni \{0, 1, 2, \dots N\}$. Then the intensity matrix of the process is formed as

$$Q = j \begin{array}{c|cccccc|} & 0 & 1 & 2 & \cdots & N-1 & N \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \mu_1 & -\mu_1 & 0 & \cdots & 0 & 0 \\ 2 & \cancel{0}\,0 & +\mu_2 & -\mu_2 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ N & 0 & 0 & 0 & \cdots & \mu_N & -\mu_N \end{array}$$

with k above the column labels.

The process $\{x(t)\}$ stares that in the state 'j' for a random length of time, whose distn. is exponential with mean $1/\mu_j$, then it moves to the state $(j-1)$; $j = 1, 2, \ldots N$. This transition continues until the state '0' is reached. Since the 1^{st} row of the matrix Q is $(0, 0, \ldots 0)$. Then the ~~[illegible]~~ state '0' is called Absorbing state.

The forward kolmogorov partial diff. equs are obtained as

$$\boxed{P_j'(t) = -\mu_j P_j(t) + \mu_{j+1} P_{j+1}(t) \; ; j = 1, 2, \ldots N-1}$$

$$\boxed{\begin{array}{ll} P_N'(t) = -\mu_N P_N(t) & ; j = N \\ P_0'(t) = \mu_1 P_1(t) & ; j = 0 \end{array}}$$

The above set of diff. equs are solved subject to the initial condition

$$P_j(0) = \begin{cases} 1 & \text{if } j = N \\ 0 & \text{if } j \neq N \end{cases}$$

It may be solved by using iterative method, the soln is obtained by

$$P_n(t) = e^{-\mu_n(t)}$$

Linear Birth and Death Process.

Let us consider the Kolmogorov forward partial diff. eqns. for the Birth and Death process as

$$p_j'(t) = -(\lambda_j + \mu_j)\, p_j(t) + \lambda_{j-1} P_{j-1}(t) + \mu_{j+1} P_{j+1}(t);$$

$$\forall\ t \geq 1 \text{ ——— } ①$$

at $j = 0$

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

at $j = n$, the eqn ① becomes

$$P_n'(t) = -(\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t)$$

when $\lambda_n = n\lambda$, $\mu_n = n\mu$

$$P_n'(t) = -(n\lambda + n\mu) P_n(t) + \lambda(n-1) P_{n-1}(t) + \mu(n+1) P_{n+1}(t)$$

$$\forall\ n \geq 1 \text{ — } ②$$

at $n = 0$

$$P_0'(t) = -(0.\lambda + 0.\mu) P_0(t) + \lambda(0-1) P_{0-1}(t) + \mu(0+1) P_{0+1}(t)$$

$$\Rightarrow P_0'(t) = \mu P_1(t)$$

By using method of prob. generating fn. eqn ② can be obtained

$$P^*(s,t) = \sum_{n=0}^{\infty} P_n(t)\, s^n \qquad \{PGF \text{ of } P_n(t)\}$$

$$\frac{\partial P^*}{\partial s} = \sum n s^{n-1} P_n(t).$$

By multiplying ② by s^n and summing over all values of $n = 1, 2, \ldots$ we get

$$\frac{\partial P}{\partial t} = -1(\lambda+\mu)\sum_{n=1}^{\infty} n P_n(t)\, s^n + \lambda \sum_{n=1}^{\infty}(n-1) P_{n-1}(t)\, s^n +$$

$$\mu \sum_{n=1}^{\infty}(n+1) P_{n+1}(t)\, s^n$$

$$= -(\lambda+\mu)s\left[\sum n P_n(t)\, s^{n-1}\right] + \lambda s^2\left[\sum (n-1) P_{n-1}(t)\, s^{n-2}\right]$$

$$+\mu\left[\sum (n+1) P_{n+1}(t)\, s^n\right]$$

$$= -(\lambda+\mu)s\frac{\partial P^*}{\partial s} + \lambda s^2\frac{\partial P^*}{\partial s} + \mu\frac{\partial P^*}{\partial s}$$

$$\frac{\partial P}{\partial t} = \left[\mu - (\lambda+\mu)s + \lambda s^2\right]\frac{\partial P^*}{\partial s} \quad \text{——} \ ③$$

where P^* is a p.g.f.

The eqn ③ is solved subject to the initial condition

$$P^*(s,0) = s^i \quad \text{——} \ ④$$

The final soln for $P^*(s,t)$ is obtained as $P^{**}(s,t)$ and it satisfies the eqn ③.

Ex: M/M/s (Queue)

In this model, the no. of service channels 's' and queueing discipline FIFO arrivals at queue follows poisson process with rate λ.

Interarrival of the customer joining the queue is exponential with parameter $1/\mu$. Service time distn. of all the customers and for all servers is same and it is distributed as exponential with mean $1/\mu$ and the service rate is μ per server.

If $x(t) \le s$, all $x(t)$ persons are being served, and departure rate $\mu x(t)$.

If $x(t) > s$, the departure rate is $s\mu$.

An account of the assumption of the exponential distn. for inter arrival time and service time are mutually indep.

This process $\{x(t), t \ge 1\}$ is a M.P of Birth & Death trial with $\lambda_n = \lambda$ and

$$\mu_n = \left.\begin{matrix} n\mu \; ; n \le s \\ s\mu \; ; n > s \end{matrix}\right\} \text{———} ①.$$

using ① in the expression

$$P_n = \left(\frac{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}}{\mu_1, \mu_2, \ldots, \mu_n} \right) P_0 \quad , \text{ we get}$$

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 \; ; & \text{if } n \le s \\ \left(\frac{\lambda}{\mu}\right)^n \frac{1}{s! \, s^{n-s}} P_0 \; ; & \text{if } n > s \end{cases}$$

P_0 can be obtained by using $\sum P_n = 1$

such that

$$P_0^{-1} = 1 + \sum_{n=1}^{s} \left(\frac{\lambda}{\mu}\right) \frac{1}{n!} + \sum_{n=s+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{s!} \frac{1}{s^{k-s}}$$

$\frac{\lambda}{s\mu} < 1$ is necessary condition for the existence of the steady state for the queueing model M/M/s.

If $\left(\frac{\lambda}{s\mu}\right) = 1$, the queue size increased infinitely and hence there is no stability in the queueing system.

Ex: when $s = 1$ (ie) M/M/1 - FIFO:

S.T a single server queueing system with $\rho < 1$ has a stationary distn. of the queue size

$$P_n = (1-\rho)\rho^n$$

Soln

when $s = 1$, in M/M/s queueing system is obtained, M/M/1 queueing system is obtained.

In this case the process $x(t)$ will be a B & D process with $\lambda_n = \lambda$, $\mu_n = \mu$ ∀ $n = 1, 2, \ldots$

and $\mu_0 = 0$, $\lambda_0 = \lambda$ for $n = 0$

Let $P_n(t) = P\{x(t) = n / x(0) = .\}$

Then the kolmogorov forward partial diff. equ of B & D process be

$$P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \quad n \geq 1 \quad \text{—— ①}$$

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \quad \text{—— ②} \quad \left\{ \begin{array}{l} \because \lambda = \lambda_0 \\ \mu = \mu_0 = 0 \end{array} \right.$$

Let us assuming the existence of steady state as $t \to \infty$

$$P_n(t) \longrightarrow \lim_n P_n, \quad \text{inde. of } t.$$

The steady state prob.

$P\{x = n\} = P_n$, can be obtained by using

Balance equ.

$P_n'(t) = 0$ and $P_n(t) = P_n$ in ① & ② we get,

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}, \quad n \geq 1 \quad \text{—— ③}$$

$$0 = -\lambda P_0 + \mu P_1 \longrightarrow ④, \quad n = 0$$

The eqn ③ & ④ are called as balance eqn of equilibrium equ.

from ④ we get

$$\mu P_1 = \lambda P_0$$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_1 = \rho P_0 \quad \left\{ \because \rho = \frac{\lambda}{\mu} \right\}$$

$\Rightarrow P_0 = \rho P_1$ $\left\{ \begin{aligned} P_0 &= \rho P_1 \\ &= \rho(\rho P_0) \\ &= \rho^2 P_0 \end{aligned} \right.$

$\vdots$

$$P_n = \rho P_{n-1}$$

$$= \rho(\rho^{n-1} P_0) \quad \left\{ \because P_{n-1} = \rho^{n-1} P_0 \right.$$

$$P_n = \rho^n P_0 \text{ ——— } ⑤$$

From eqn ③,

$$0 = -(\lambda P_n + \mu P_n) + \lambda P_{n-1} + \mu P_{n+1}$$

$$0 = -\lambda P_n - \mu P_n + \lambda P_{n-1} + \mu P_{n+1}$$

$$\mu P_{n+1} - \lambda P_n = \mu P_n - \lambda P_{n-1}$$

$\div \mu$ on both sides,

$$P_{n+1} - \left(\frac{\lambda}{\mu}\right) P_n = P_n - \left(\frac{\lambda}{\mu}\right) P_{n-1}$$

$$P_{n+1} - \rho P_n = P_n - \rho P_{n-1}$$

For $n = n-1$,

$$P_{n-\cancel{1}+\cancel{1}} - \rho P_{n-1} = P_{n-1} - \rho P_{n-1-1}$$

$$P_n - \rho P_{n-1} = P_{n-1} - \rho P_{n-2}$$

For $n = n-2$

$$P_{n-2+1} - \rho P_{n-2} = P_{n-2} - \rho P_{n-2-1}$$

$$P_{n-1} - \rho P_{n-2} = P_{n-2} - \rho P_{n-3}$$

$$\therefore P_n - \rho P_{n-1} = P_{n-1} - \rho P_{n-2}$$

$$= P_{n-2} - \rho P_{n-3}$$

$$\vdots$$

$$= P_1 - \rho P_0$$

$$= 0 \quad \therefore n \geq 0$$

Since $P_n = \rho P_{n-1}$

$= \rho^n P_0$ and $\sum_{n=0}^{\infty} P_n = 1$

$$\Sigma P_0 \rho^n = P_0 (\Sigma \rho^n)$$

$$= P_0 (1 + \rho + \rho^2 + \ldots + \rho^n + \ldots)$$

$$= P_0 (1 - \rho)^{-1} = 1 \qquad \{\because P_0 = 1 - \rho$$

$$\Rightarrow P_0 = 1 - \rho$$

$$\vdots$$

$$P_n = (1 - \rho) \rho^n, \quad n \geq 0$$

$\rho = \frac{\lambda}{\mu} < 1$ The steady state solu of the process (simple queueing model exist).

The distn. $P_n = (1-\rho)\rho^n$ is of a geometric distn. Hence mean & variance of the queue size can be obtained as

$$E(x) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$$

$$V(x) = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

The Prob. that the system is empty is obtained as

$$P\{x=0\} = P_0 = 1-\rho$$

The prob/. that the system is not empty is obtained as

$$P\{x \geq 1\} = 1 - P\{x < 1\}$$

$$= 1 - P\{x = 0\}$$

$$= 1 - P_0$$

$$= 1 - (1-\rho).$$

$$P\{x \geq 1\} = \rho$$

Ex: M/M/∞ ; FIFO

It is obtained M/M/S ; FIFO as a special case of M/M/S with $S = \infty$. In real life the no. of servers can be increase so that at any time the no. of servers equal to the no. of customers.

There will be no waiting. This model is more appropriate in the large communication system with the assumptions used in M/M/S system in addition to the above assumption. The process $x(t)$ will be a M.P of B & D trial with $\lambda_n = \lambda$, $\mu_n = n\mu \; \forall \; n = 0, 1, ..$

eqn ① is substituted in

$$P_n = \left[\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right] P_0$$

Hence we get,

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 \; ; \; \forall \; n = 1, 2, \cdots \quad \text{—— ②}$$

$$\Sigma P_n = 1 \Rightarrow P_0 \sum_n \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} = 1$$

$$P_0 \Sigma \frac{\rho^n}{n!} = 1 \; ; \quad \text{where } \rho = \frac{\lambda}{\mu}$$

$$P_0 (e^\rho) = 1 \Rightarrow P_0 = e^{-\rho} \quad \left\{ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \right.$$

$$\therefore P_n = \frac{e^{-\rho} \rho^n}{n!}$$

Thus the stable distn. of M/M/∞ exist whatever be the value of ρ and it is of the poisson distn. with parameter ρ. The steady state soln of this model exist if the series $\Sigma \frac{\rho^n}{n!}$ convergent.